# Matrix-Ball Construction for affine Robinson-Schensted correspondence

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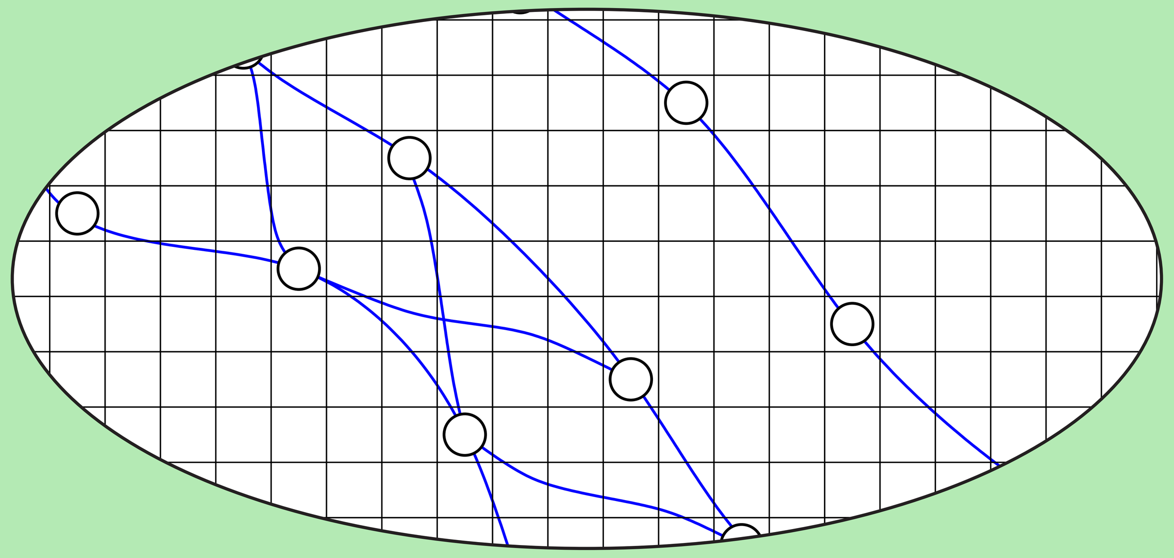


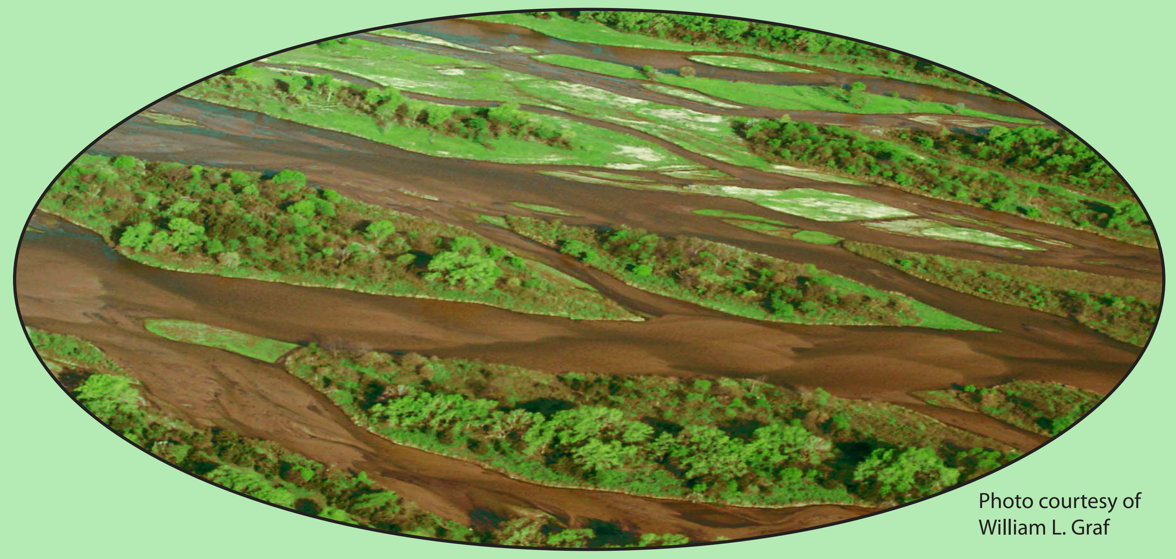
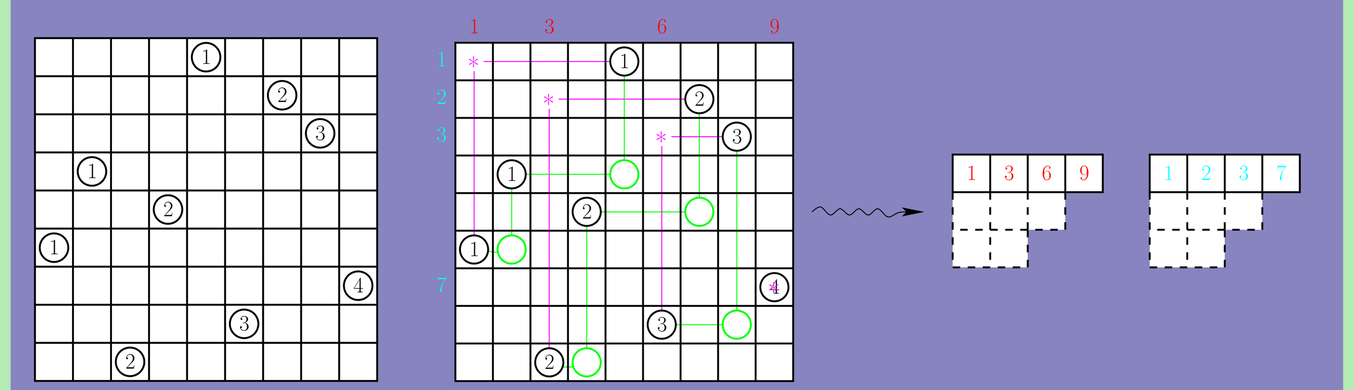


Photo courtesy of William L. Graf

## "Recall": Matrix-Ball Construction

$w = 578241963$

Numbering: Step:



**Full example on the computer (source code available [C])**

## Motivation: Kazhdan-Lusztig Theory

The Hecke algebra

$$\mathcal{H} = \left\langle T_s, s \in S \;\middle|\; \begin{array}{c} T_sT_tT_s\ldots = T_tT_sT_t\ldots \\ (T_s+1)(T_s-q)=0 \end{array} \right\rangle$$

of a Coxeter system $(W,S)$ has two bases $\{T_w\}_{w\in W}$ and $\{C_w\}_{w\in W}$ (the transition matrix consists of Kazhdan-Lusztig polynomials). It acts on itself by left multiplication and this action is cellular, i.e. there exist subsets $L$ of $W$ (cells) which carry subquotient representations:

$$u \in L, v_1, v_2 \in L, v_3, v_4 \notin L: \qquad T_sC_u = C_{v_1} + C_{v_2} + C_{v_3} + C_{v_4}$$

For the symmetric group the cells are given by Robinson-Schensted correspondence:

$$L_Q = \{w \in S_n \mid Q(w) = Q\}.$$

**Goal:** Find an affine generalization of some standard construction of Robinson-Schensted correspondence which describes cells similarly.

## Bijection

$W \hookrightarrow \Omega_{dom}$

Extended affine symmetric group

$\widetilde{S}_n = \{w : \mathbb{Z} \hookrightarrow \mathbb{Z} \mid w(i+n) = w(i)+n\}$

$[7,-4,3,5,9] =$

$$\begin{array}{ccccccccccccc} \cdots & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ \cdots & 0 & 4 & 7 & -4 & 3 & 5 & 9 & 12 & 1 & \cdots \end{array}$$

or affine symmetric group

$$\widetilde{S}_n = \left\{ w \in \widetilde{S}_n \;\middle|\; \sum_{i=1}^{n} w(i) - i = 0 \right\}$$

$$\Omega = \left\{ \left( \boxed{P}, \boxed{Q}, \boxed{\rho} \right) \right\} \qquad \mathbb{Z}^{\ell(\mathrm{sh}(P))}$$

tabloids of same shape filled with:

$$\overline{1} := 1 + n\mathbb{Z}, \overline{2}, \ldots, \overline{n}$$

Offset dominance:

$$\forall i, P, Q\ \exists r_i^{P,Q} \in \mathbb{Z} \cup \{-\infty\}: \quad \rho_{i+1} \geqslant \rho_i + r_i^{P,Q}$$

Non-extended: $\sum_i \rho_i = 0$

## Proper numberings

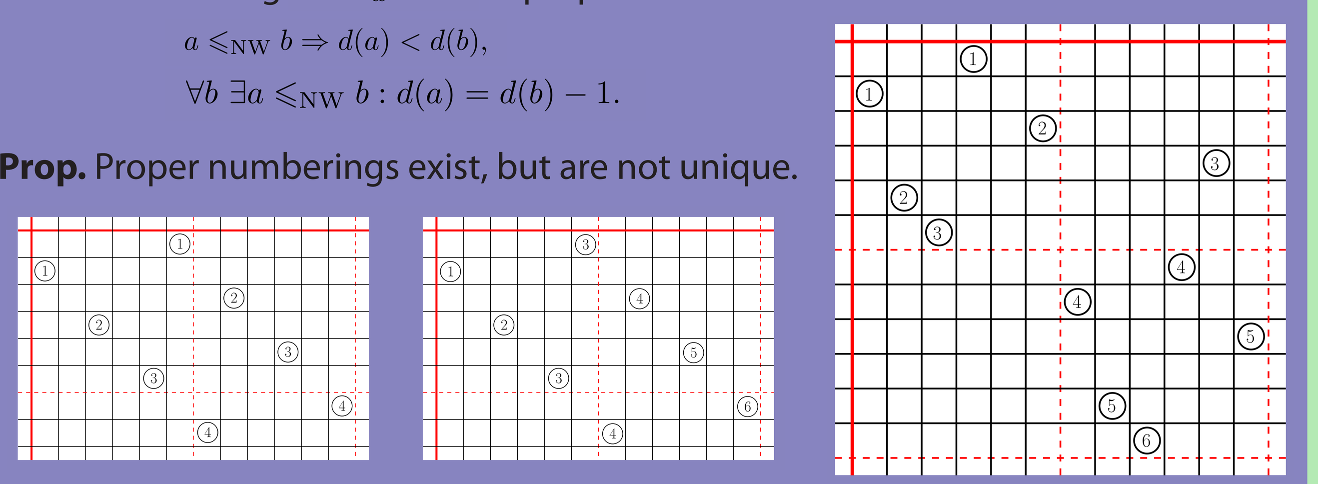
Set of balls of $w$ : $\mathcal{B}_w$

**Def.** A numbering $d : \mathcal{B}_w \to \mathbb{Z}$ is proper if

$$a \leqslant_{\mathrm{NW}} b \Rightarrow d(a) < d(b),$$
$$\forall b\ \exists a \leqslant_{\mathrm{NW}} b : d(a) = d(b) - 1.$$

**Prop.** Proper numberings exist, but are not unique.

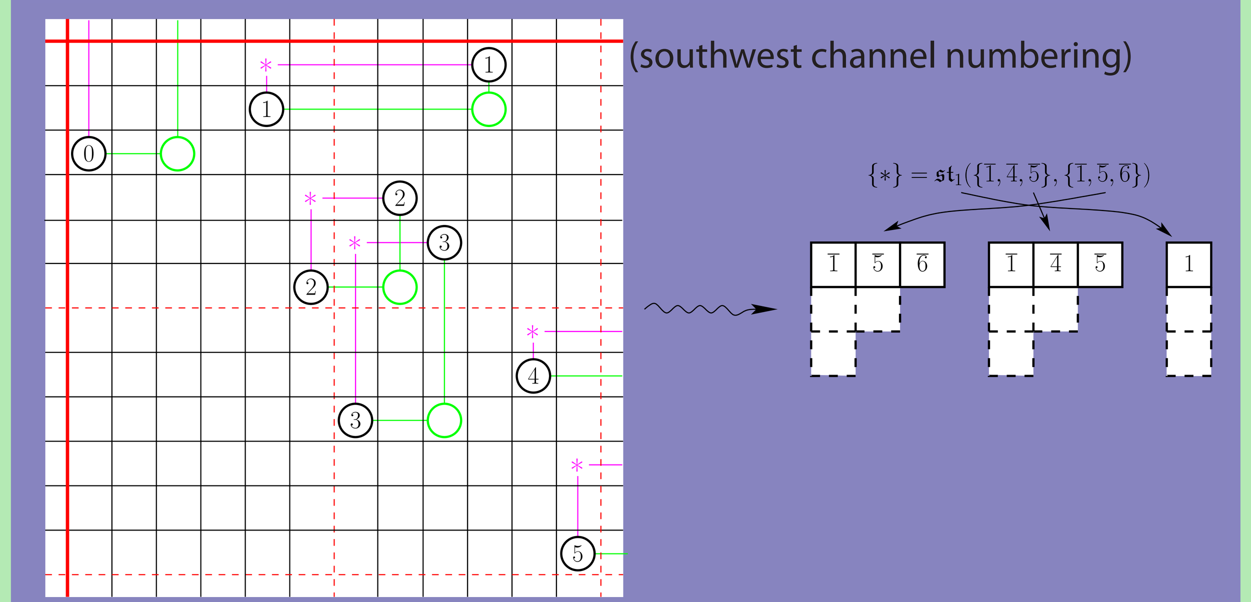
$w = [4, 1, 6, 11, 2, 3]$



**Prop.** Proper numberings are semi-periodic.

## Affine Matrix Ball Construction

(southwest channel numbering)



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## History

- Kazhdan-Lusztig theory began with the seminal paper [KL].
- Lusztig studied cells in affine Weyl groups in a series of papers [L].
- Shi continued the study of cells with emphasis on the affine symmetric group [S1]
- Shi gave an algorithm [S2] which takes

$$w \mapsto P(w)$$

such that two permutations are in the same left cell precisely when they are mapped to the same tabloid (the algorithm does not generalize a known RS alrogirthm).

**Thm.** This map is the same as taking the left tabloid in AMBC.

- Honeywill [H] extended Shi's algorithm to a bijection (introducing a version of weights).

## Constructing proper numberings

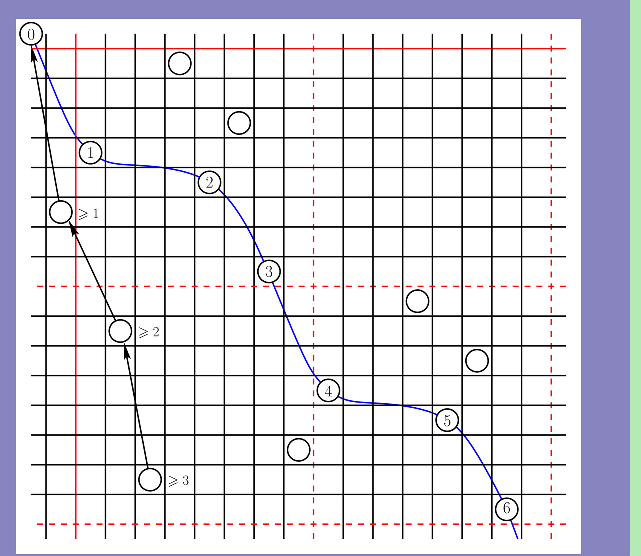
**Def.** A channel is a densest periodic collection of balls in pairwise negative slope.

**Def./Prop.** Channel numbering:

$d(b)$ = maximal lower boud

This is finite.

**Rmk.** There is no need to consider paths with two balls that are translates.

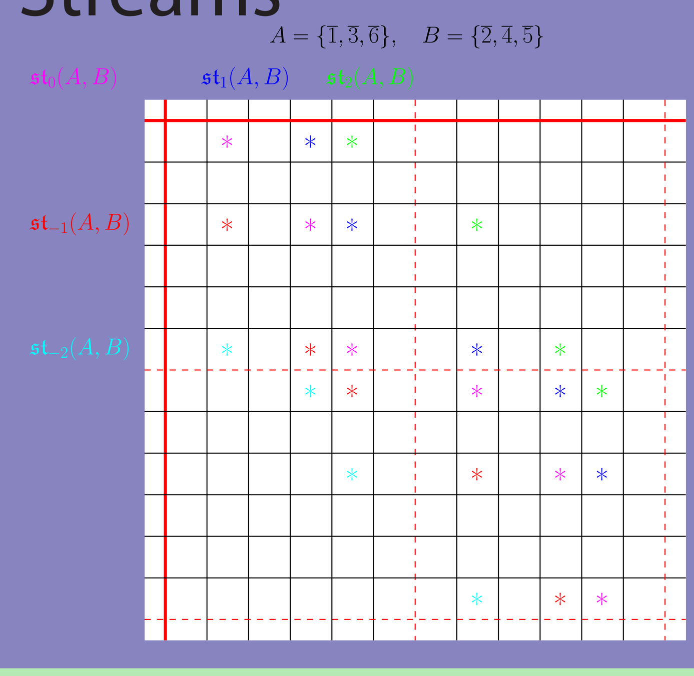


## Streams

**Def.** A stream is a periodic collection of cells in pairwise negative slope.

**Fact.** A stream is specified by:

1) set of columns,
2) set of rows,
3) an integer.




## References


[C] M. Chmutov, AMBC Java program, http://math.umn.edu/~mchmutov/affiners.html

[H] T. Honeywill, *Combinatorics and algorithms associated with the theory of Kazhdan-Lusztig cells.* Thesis. (2005)

[KL] D. Kazhdan, G. Lusztig, *Representations of Coxeter groups and Hecke algebras.*

[L] G. Lusztig, *Cells in affine Weyl groups I, II, III.*

[S1] J. Shi, *The Kazhdan-Lusztig cells in certain affine Weyl groups.*

[S2] J. Shi, *The generalized Robinson-Schensted algorithm on the affine Weyl group of type* $A_{n-1}$ .