Affine Robinson-Schensted correspondence in Kazhdan-Lusztig theory

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Brunswick, ME

Spetember 24, 2016

Affine Matrix-Ball Construction [CPY]

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- Left-hand side = affine symmetric group

- Right-hand side

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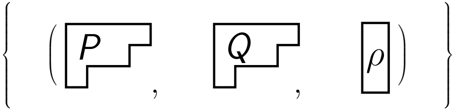
$\Omega =$ 

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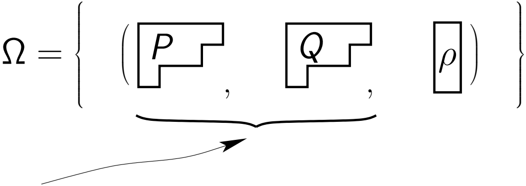

tabloids of same shape filled with $\overline{1} := 1 + n\mathbb{Z}, \overline{2}, \ldots, \overline{n}$.

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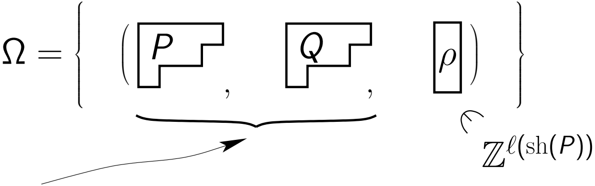

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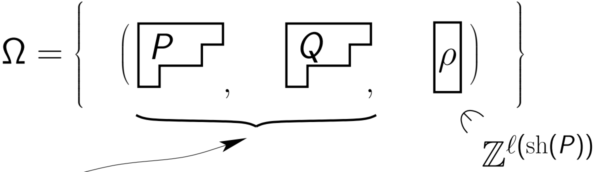

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Non-extended $\Leftrightarrow \sum_i \rho_i = 0$

Dominance, inverses, and involutions.

Offset dominance: $\forall i, P, Q$ if $\lambda_i = \lambda_{i+1}$ then $\rho_{i+1} \geqslant \rho_i + (\mathrm{ch}_i(P) - \mathrm{ch}_i(Q))$

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3
5
-7

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If $w \mapsto (P, Q, \rho)$ then $w^{-1} \mapsto (Q, P, \rho')$ and ρ' = dominant representative of $-\rho$.

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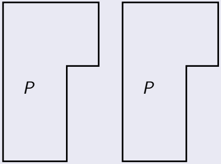
-5
-3
7

Corollary

Involutions:



-3
3
-1
0
1

Descents and signs.

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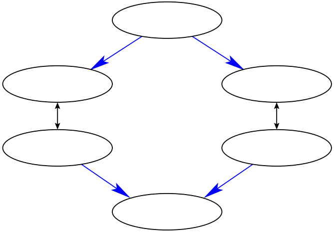

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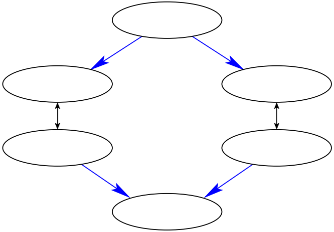

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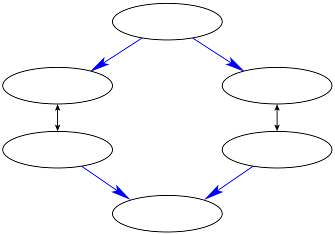

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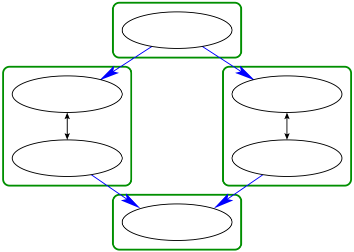

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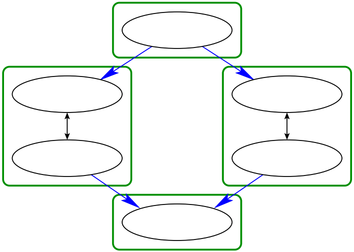

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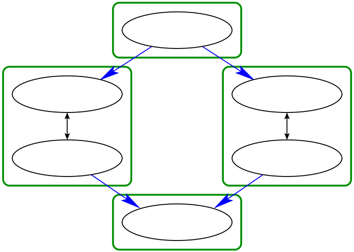

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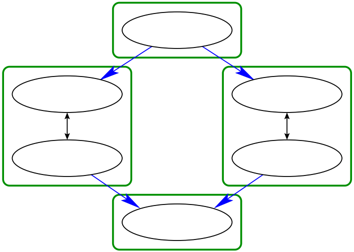
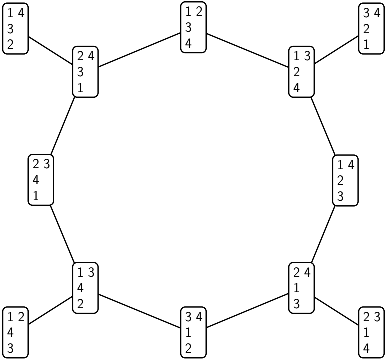


Knuth moves:

$$\ldots, 0, {\color{red}3}, [{\color{red}-3}, 6, 4, 7], 1, 10, \ldots$$

$\overline{2}$	$\overline{4}$
$\overline{3}$	
$\overline{1}$	

0
0
1



Affine Knuth equivalence classes

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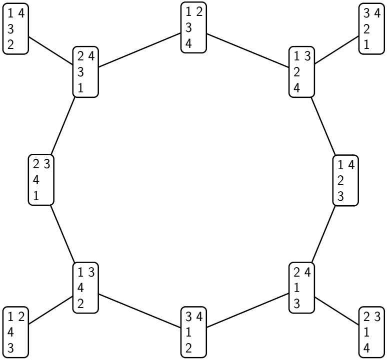
0
0



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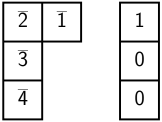
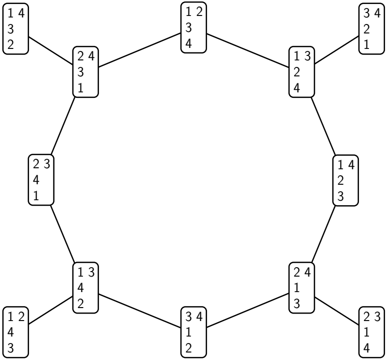


- $w \mapsto Q(W)$ - graph covering



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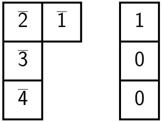

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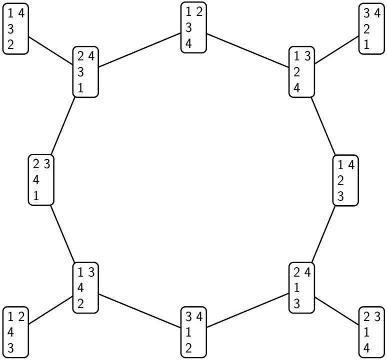

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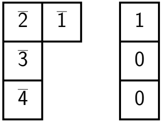

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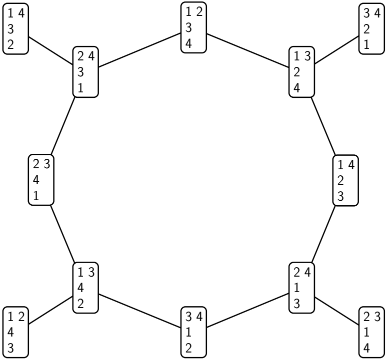

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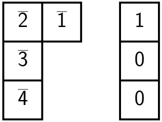

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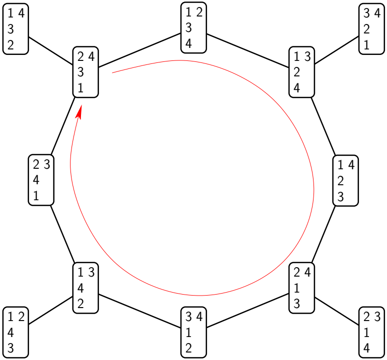
- Example:

$$\begin{pmatrix}0\\0\\1\end{pmatrix} + {\color{red}\begin{pmatrix}-2\\1\\1\end{pmatrix}} = \begin{pmatrix}-2\\1\\2\end{pmatrix}$$



Thank you!