Parallel transport in the Kazhdan-Lusztig W-graph and Green's $0-1$ conjecture in Lie type B

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The Hecke Algebra and Kazhdan-Lusztig polynomials

(W, S) – Coxeter system, ground ring: $\mathbb{Z}[q^{\pm 1/2}]$

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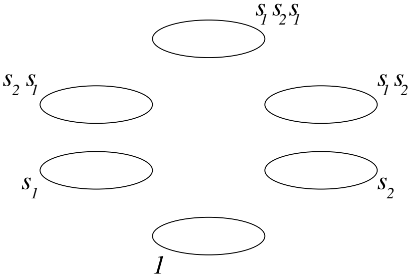
- $deg(P_{v,w}) \leqslant \frac{l(w)-l(v)-1}{2}$; $\mu(v, w) = \left[q^{\frac{l(w)-l(v)-1}{2}} \right] P_{v,w}$ (symmetrized)

- Vertices: elements of W



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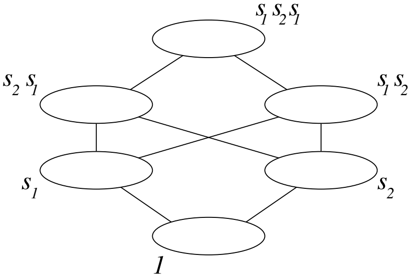

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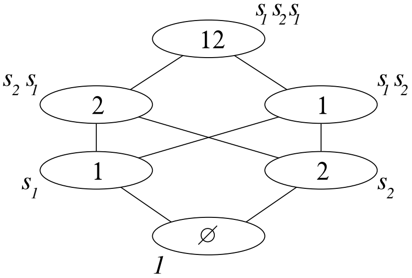

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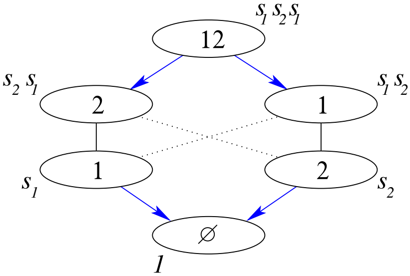

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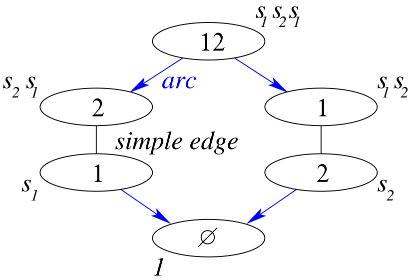

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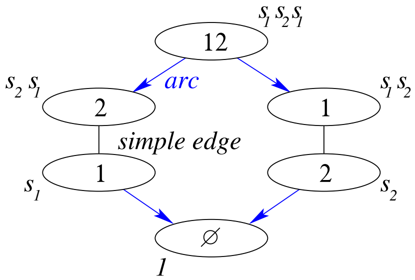

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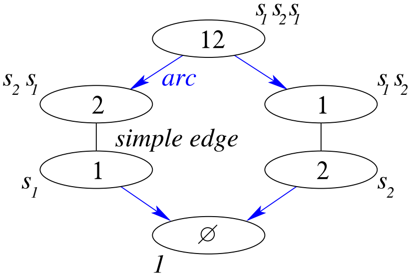

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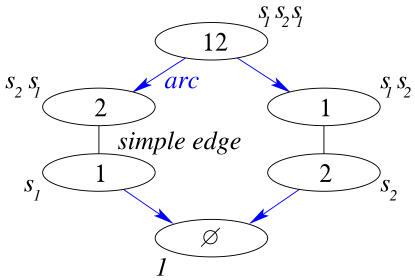

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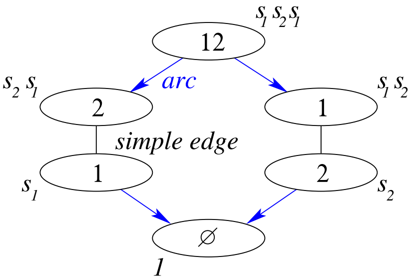

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Green's $0 - 1$ Conj. A, $\widetilde{A}$ (Green, 2009); D (Gern, 2013); B (C., 2014)

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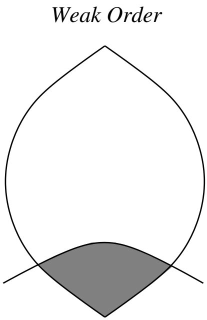
Any two reduced expressions of w are connected by braid moves.



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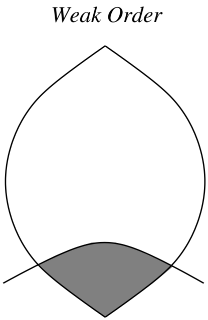

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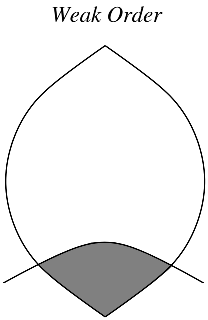

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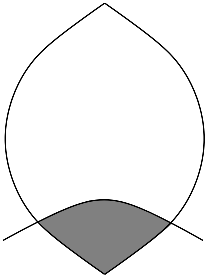

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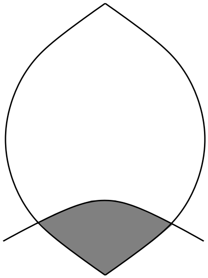

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- τ-labels do not contain adjacent simple reflections.

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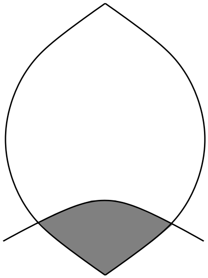

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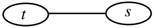

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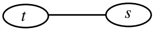

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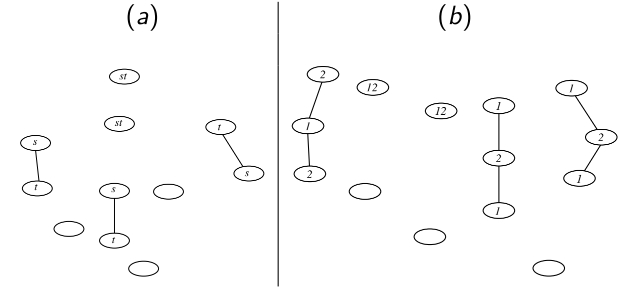

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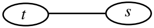

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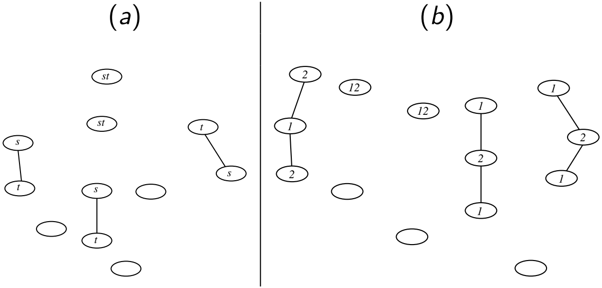
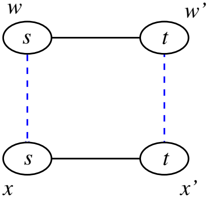
- If $order(s_1 s_2) = 4$, graph restricted to parabolic $\langle s_1, s_2 \rangle$ is in (b).



Parallel transport in simply laced types

For s, t with $order(st) = 3$ and



we have $\mu(x, w) = \mu(x', w')$.

Example: A_5

Length 1 edges (in Bruhat order) have weights 0 or 1.

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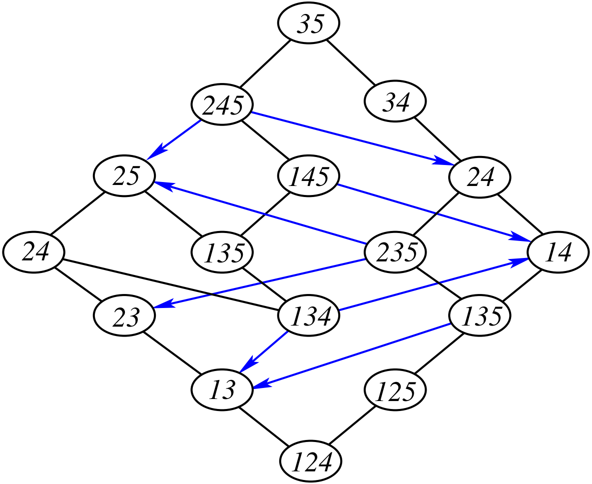

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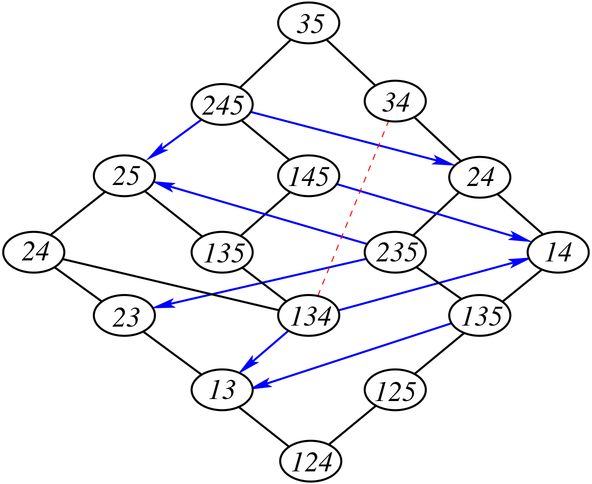

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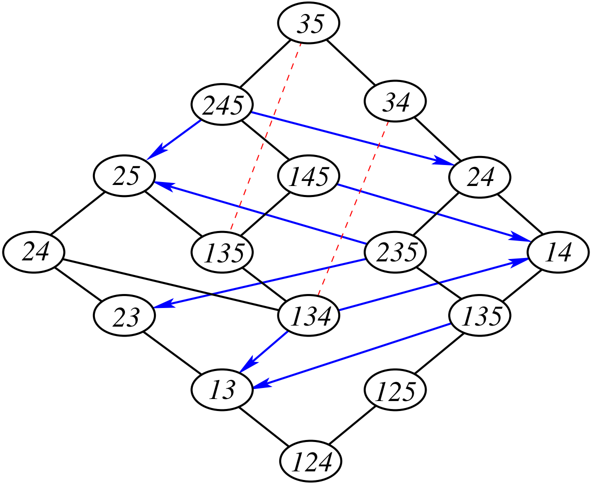

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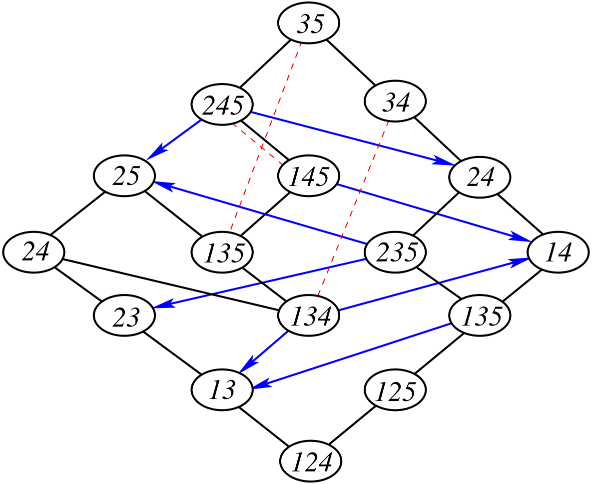

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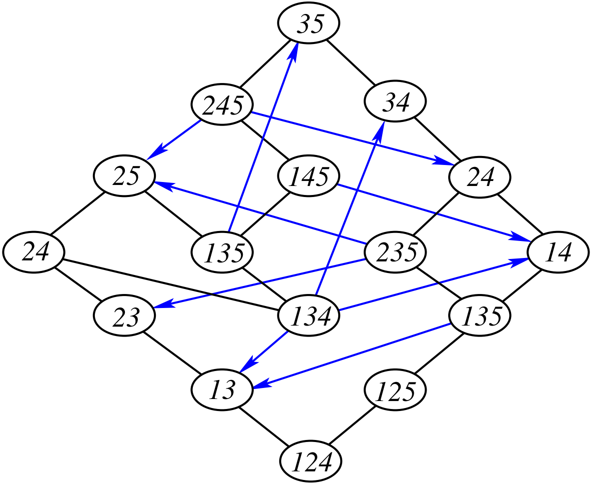

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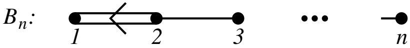

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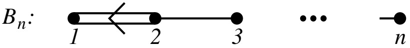
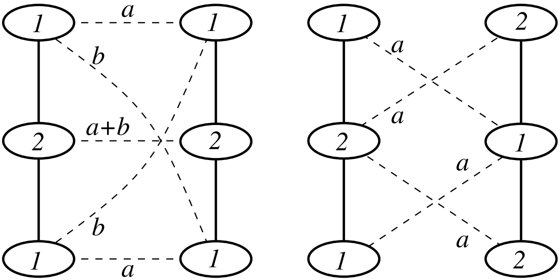


Additional relations for the double bond:



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Type B obstruction to pulling down: simple edges activating (1,2). Fix them using parallel transport and pattern-avoidance characterization.

Thank you!