Type A molecules are Kazhdan-Lusztig

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University of Michigan

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FPSAC '13

The Iwahori-Hecke Algebra and Kazhdan-Lusztig polynomials

$W = S_n$, ground ring: $\mathbb{Z}[q^{\pm 1/2}]$

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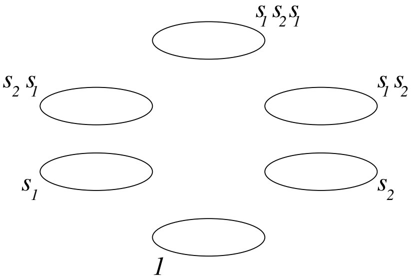
- $deg(P_{v,w}) \leqslant \frac{l(w)-l(v)-1}{2}$; $\mu(v,w) = \left[q^{\frac{l(w)-l(v)-1}{2}} \right] P_{v,w}$

(Example) Kazhdan-Lusztig W-graph

- Vertices: elements of S_n



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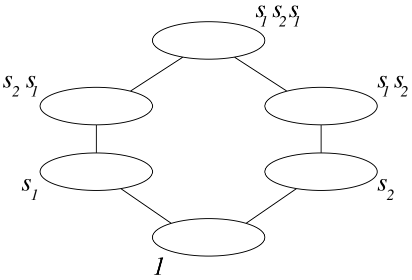

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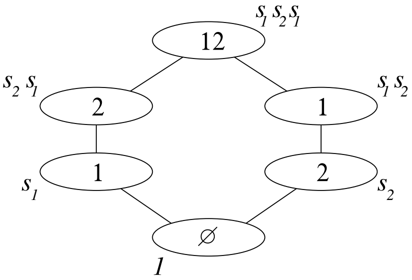

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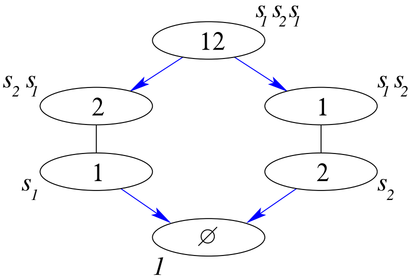

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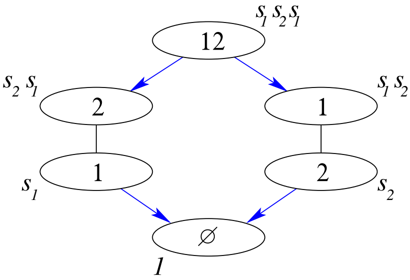

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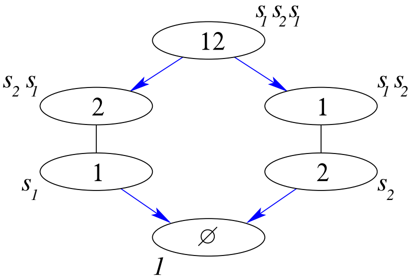

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General Goal

Get your hands on (subgraphs of) KL graph without computing KL polynomials.

Outline

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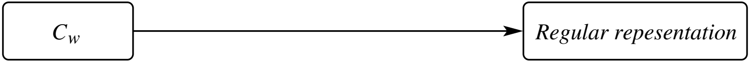

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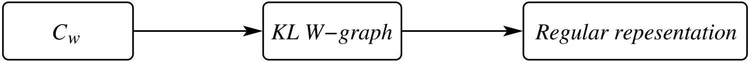

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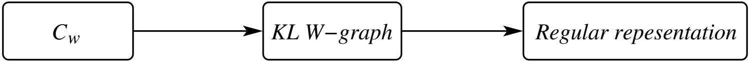

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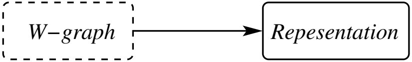

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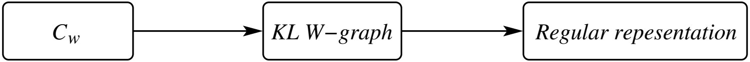

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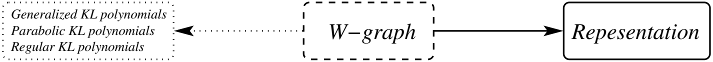
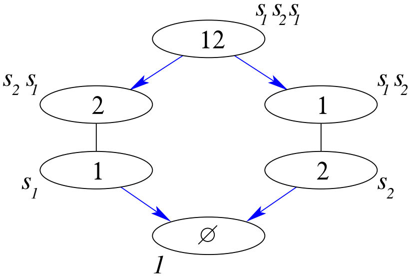


From graph to representation

Basis: vertices.

$$T_i u = \begin{cases} qu & i \notin \tau(u) \\ -u + q^{1/2} \displaystyle\sum_{\substack{u \to v \\ i \notin \tau(v)}} m(u \to v) v & i \in \tau(u) \end{cases}$$



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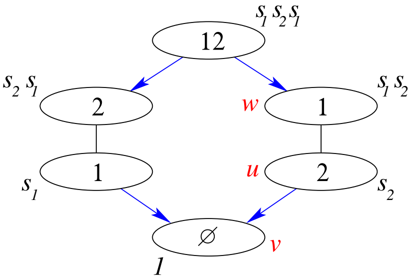

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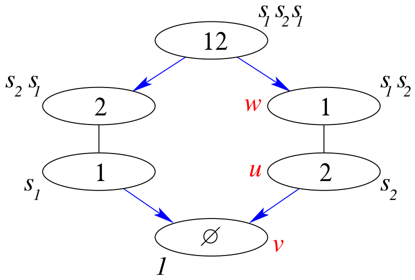

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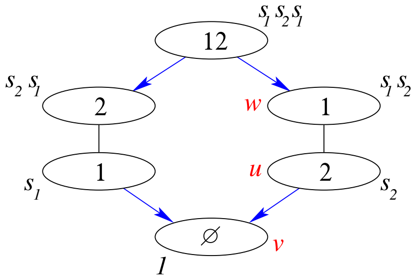

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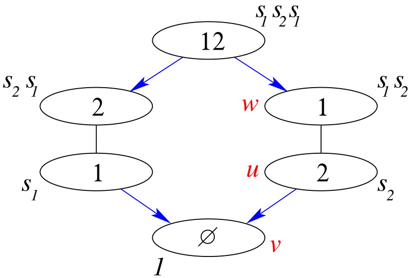
Remark

This is how T_i's act on the C_w basis with respect to KL graph.



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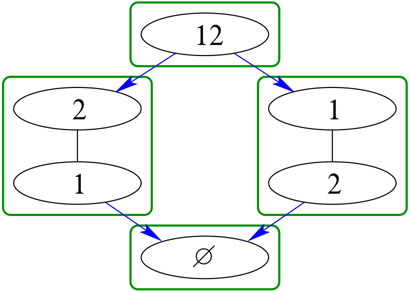

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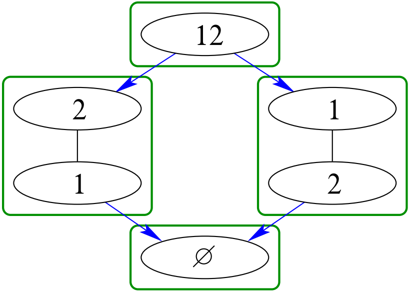

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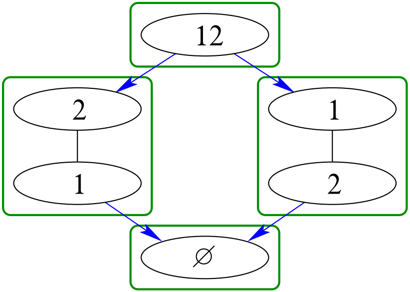

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- A cell is a W-graph. (Why?)
- How many cells are there?
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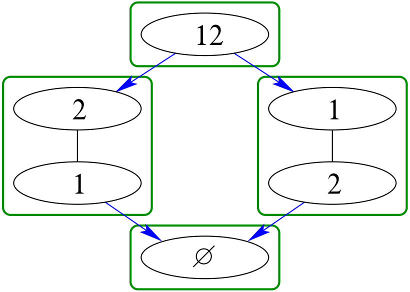
- Do all S_n cells come from the KL graph? (Up to $n = 13\ldots$)



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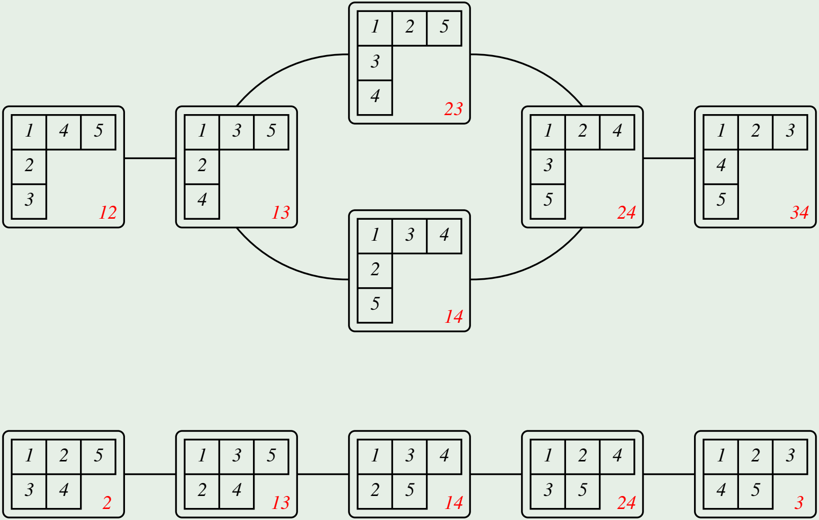
- Cells with Qs of same shape are isomorphic
- *Simple edges*, i.e. edges going in both directions, are dual Knuth moves

Simple edges in Kazhdan-Lusztig Cells

Examples



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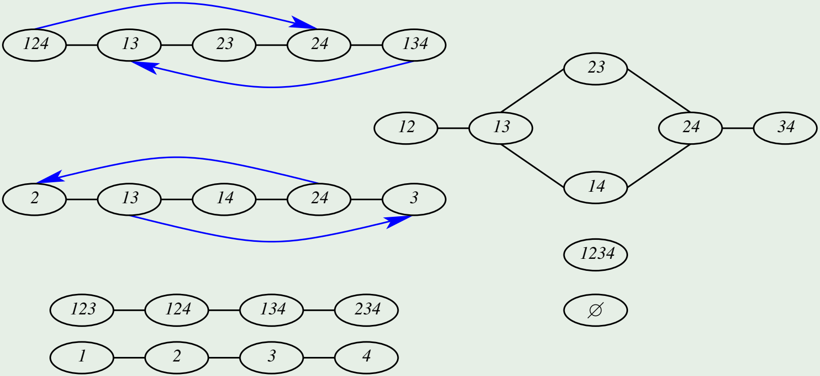
Classification of S_5 cells



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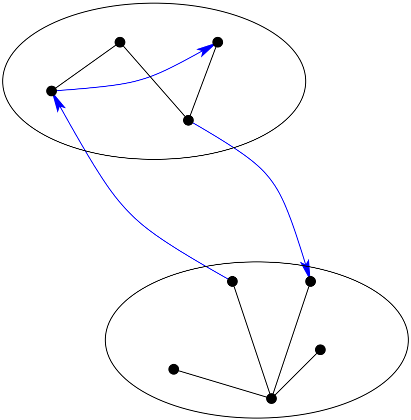

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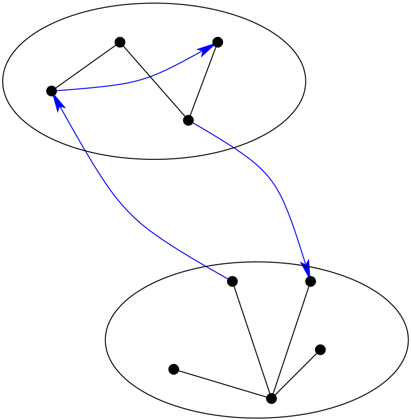
Fact

Each Kazhdan-Lusztig S_n cell has only one molecular component.



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- Other types?

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DEG: molecular component of KL cell; viewed as undirected graph.

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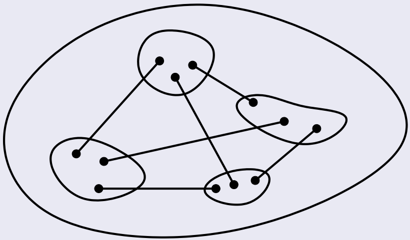
Axiom 6; in molecular language



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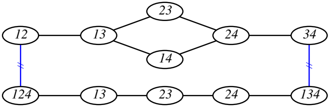

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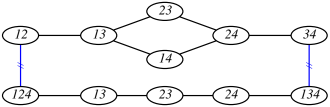

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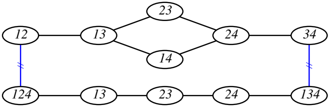

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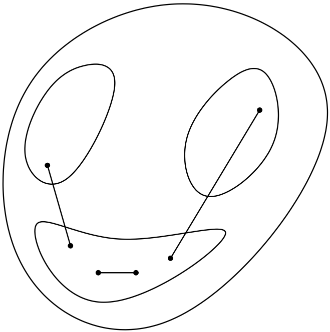

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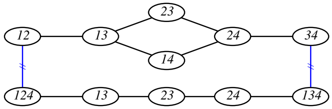

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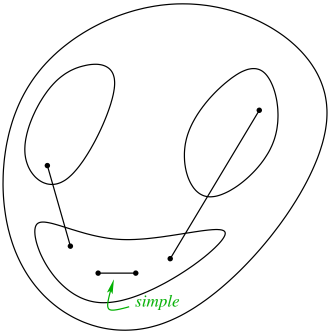

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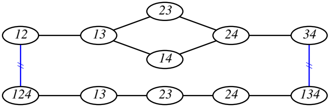

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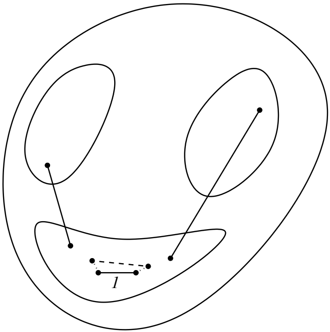

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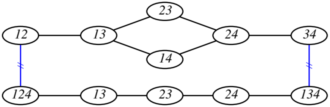

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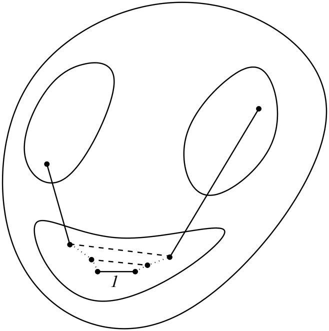

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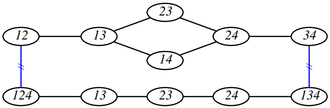

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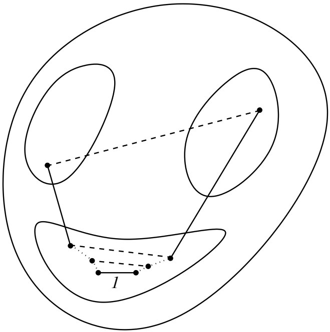

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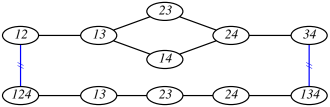
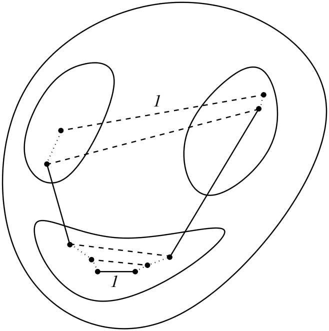


- Prove that if axiom 6 does not hold then restrictions are inconsistent.



Thank you!