Type A molecules are Kazhdan-Lusztig

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April 20, 2013

The Iwahori-Hecke Algebra and Kazhdan-Lusztig polynomials

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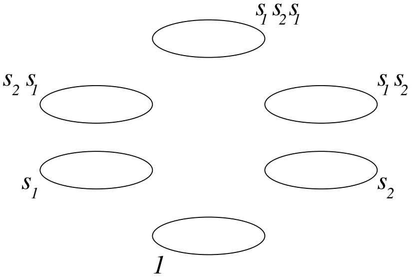
- $deg(P_{v,w}) \leqslant \frac{l(w)-l(v)-1}{2}$; $\mu(v,w) = \left[q^{\frac{l(w)-l(v)-1}{2}}\right] P_{v,w}$

Kazhdan-Lusztig W-graph

- Vertices: elements of S_n



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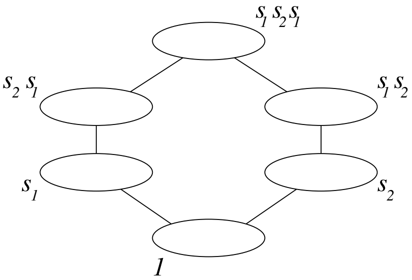

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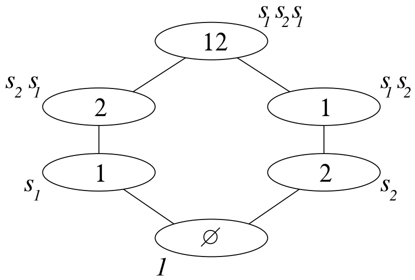

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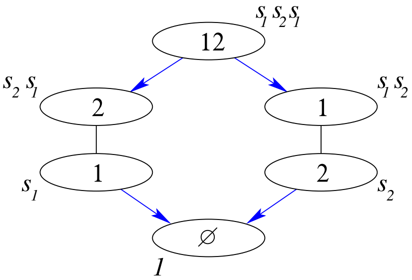

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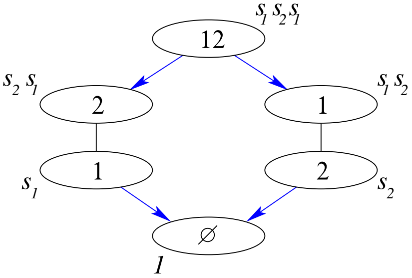


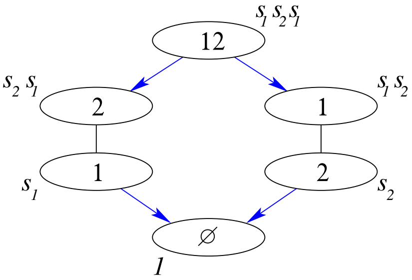
Fact

It is easy to reconstruct Kazhan-Lusztig polynomials once one has the W-graph.

From graph to representation

Basis: vertices.

$$T_i u = \begin{cases} qu & i \notin \tau(u) \\ -u + q^{1/2} \sum_{\substack{u \to v \\ i \notin \tau(v)}} m(u \to v) v & i \in \tau(u) \end{cases}$$



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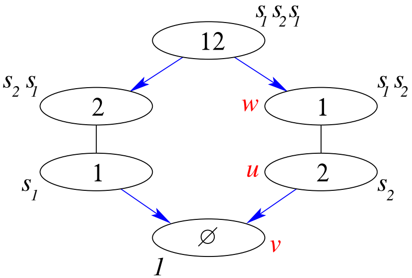

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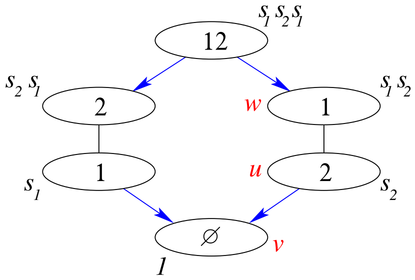

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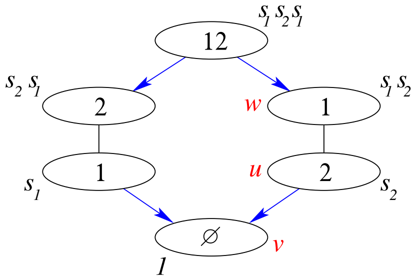

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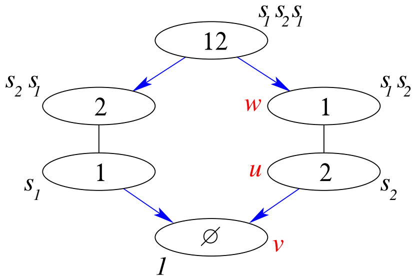
Remark

This is how T_i's act on the C_w basis with respect to KL graph.



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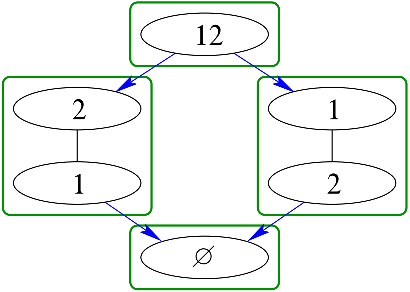

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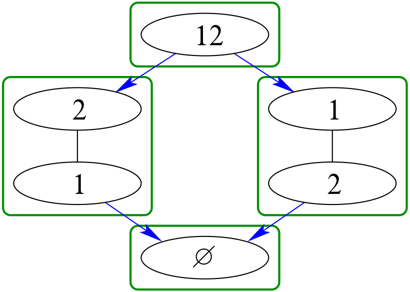

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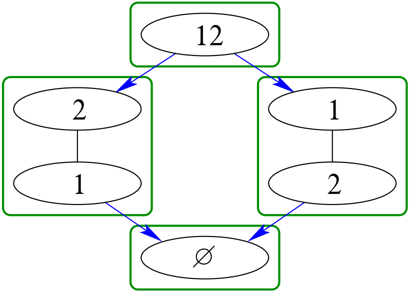

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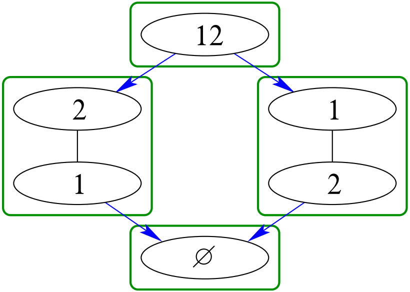

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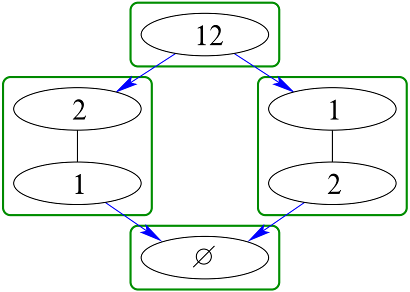

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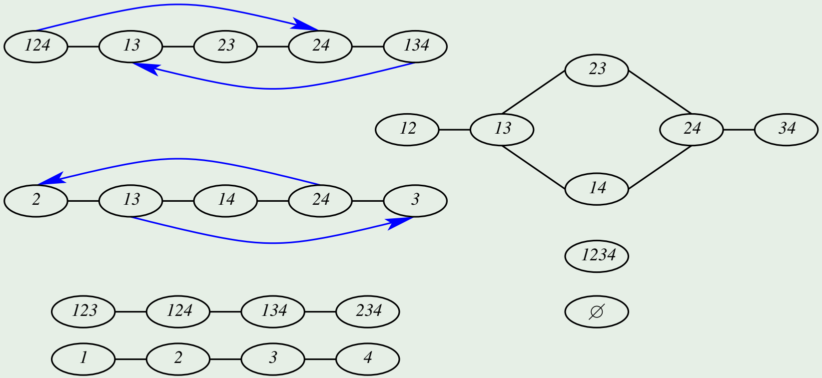
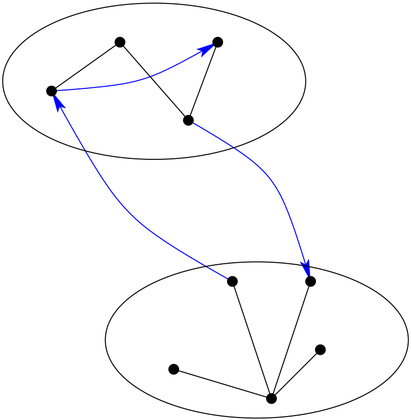
Classification of S_5 cells



Molecules

Molecule:



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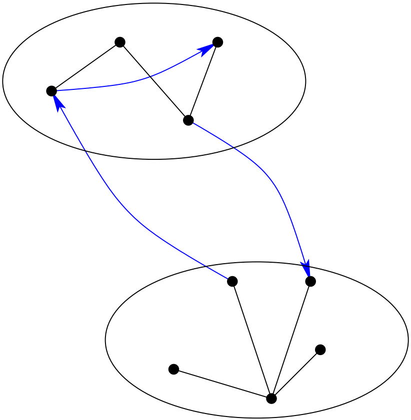

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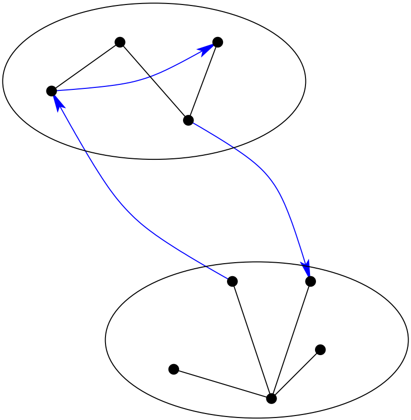

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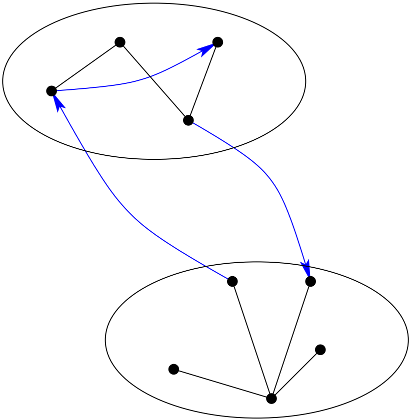

Fact

Each simple edge component of a W-graph is a molecule.

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Kazhdan-Lusztig Molecules

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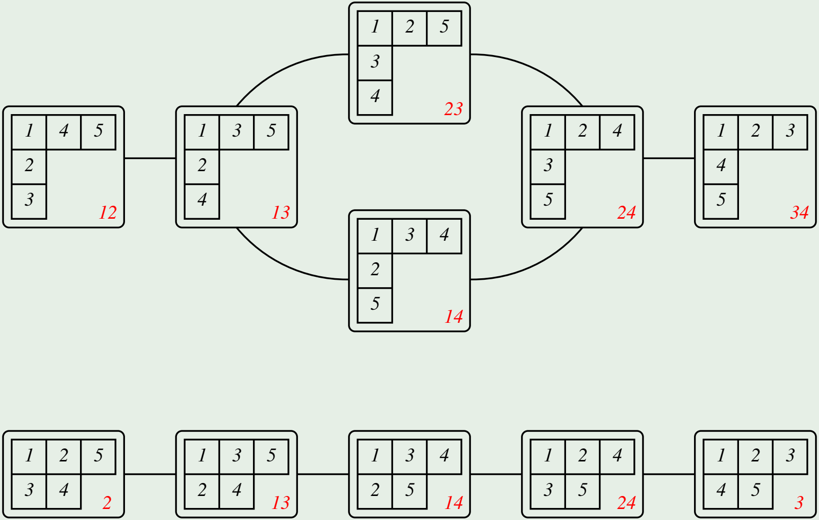

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