

## UMTYMP Advanced Topics; Fall 2016

### Homework 10 (due 11/15/16)

**Problem 1.** Let  $a_n$  be the number of ways to pay  $n$  dollars using ten-dollar, five-dollar, and one-dollar bills. Find the ordinary generating function  $\sum_{n \geq 0} a_n x^n$ .

**Problem 2.** Fix  $n \in \mathbb{Z}^{>0}$ . In how many ways can one choose a composition  $\alpha$  of  $n$ , and then choose a composition of each part of  $\alpha$ ?

**Problem 3.** A line of snakes is waiting for a bus. The length of the line is  $n$  meters. Each snake is  $k$  meters long for some  $k \geq 2$ . A snake that is  $k$  meters long has  $k - 2$  spots, each either black or orange. Also, each snake is either very poisonous, slightly poisonous, or not poisonous at all. Find a simple formula for the number  $f(n)$  of possibilities. For instance,  $f(1) = 0$  (since no snake is one meter long),  $f(2) = 3$  (one snake of length 2 meters with no spots and three possible levels of poisonosity).

**Problem 4.** Let  $h_n$  be the number of ways to tile a  $1 \times n$  rectangles  $1 \times 1$  tiles that are red or blue and  $1 \times 2$  tiles that are green, yellow, or white. Find a closed formula for  $H(x) = \sum_{n \geq 0} h_n x^n$ .

**Problem 5.** Let  $a_{n,k}$  be the number of weak compositions of  $n$  into  $k$  parts.

- (1) Use one of the “Rules” from class to find the generating function  $A_k := \sum_{n \geq 0} a_{n,k} x^n$ .
- (2) The power series  $A_k$  above is a product of some FPS’s. Explain the bijection between the set of weak compositions  $S = \{(n_1, n_2, \dots, n_k) \mid n_i \in \mathbb{Z}^{\geq 0}, n_1 + n_2 + \dots + n_k = n\}$  and the set of terms in the expansion of the product. (If you have trouble working in the general case, do some examples first.)
- (3) Let  $b_{n,k}$  be the number of compositions of  $n$  into  $k$  parts, each one being 2 or 3. Find the generating function  $B_k := \sum_{n \geq 0} b_{n,k} x^n$ .