UMTYMP Advanced Topics; Fall 2016
Homework 1 solutions

Problem 1. (1) *In April, Ms. Consistent went to the swimming pool on 26 days. Is it necessarily true that she went to the swimming pool on six consecutive days?*
(2) *What if the month was May?*

Solution. It is clear that the longest time period when she does not go to the pool on six consecutive days, assuming she takes k days off, is $k + 5(k + 1)$. In April she must have taken only 4 days off, so she could have gone without visiting the pool six days in a row only for $4 + 5(4 + 1) = 29$ days. Thus the statement is necessarily true. In May she must have taken 5 days off, so she could have gone without visiting the pool six days in a row for $5 + 5(5 + 1) = 35$ days. Thus the statement is not necessarily true.

Problem 2. *Prove that every year contains at least four and at most five months that contain five Sundays.*

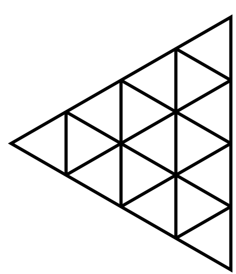
Solution. A period of k days has at least $\lfloor k/7 \rfloor$ Sundays (if it starts on a Monday) and at most $\lfloor k/7 \rfloor + 1$ Sunday (if it starts on a Sunday). Since there are between 28 and 31 days in a month, every month contains either 4 or 5 Sundays. Similarly, a year has either 365 or 366 days, so a year has either 52 or 53 Sundays. Now $52 = 4 \cdot 12 + 4$, so a year containing 52 Sundays has four months that have 5 Sundays. Similarly, a year containing 53 Sundays has five months that have 5 Sundays.

Problem 3. *Prove that there exists a positive integer n such that $44^n - 1$ is divisible by 7.*

Solution. By pigeonhole principle, for some distinct k and l we have $44^k \equiv 44^l \pmod 7$; upon renaming if necessary, we may assume $k < l$. So $44^l - 44^k$ is divisible by 7. Now $44^l - 44^k = 44^k(44^{l-k} - 1)$. Since 44 is relatively prime to 7, we conclude that $44^{l-k} - 1$ must be divisible by 7.

Problem 4. *We are given 17 points inside a regular triangle of side length 1. Prove that the distance between some pair of these points is at most 1/4.*

Solution. Subdivide the triangle as shown in the figure below. Since there are 16 small triangles, by pigeonhole principle, one of them contains at least two of the points. The side of each small triangle is 1/4, so the distance between the two points in the small triangle is at most 1/4.



Problem 5. *Let α be any irrational real number. Prove that there exists a positive integer n such that the distance from $n\alpha$ to the closest integer is less than 10^{-17}.*

Solution. For any n, let $f(n)$ denote the fractional part of $n\alpha$ (i.e. $f(n) = n\alpha - \lfloor n\alpha \rfloor$). Note that if $m \neq n$ then $f(m) \neq f(n)$ since otherwise α would be rational. Split the half-open interval $[0, 1)$ into 10^{17} subintervals of equal length (so the length of each subinterval is 10^{-17}). Since f takes infinitely many distinct values, by pigeonhole principle, there exist positive integers $k < l$ such that $f(k)$ and $f(l)$ fall in the same subinterval, i.e. $|f(l) - f(k)| < 10^{17}$. Now, $(l - k)\alpha = \lfloor l\alpha \rfloor - \lfloor k\alpha \rfloor + f(l) - f(k)$ is within 10^{17} of $\lfloor l\alpha \rfloor - \lfloor k\alpha \rfloor$.

Problem 6. *Suppose we have* 1010 *positive integers. Prove that we can find a pair such that either the sum or the difference is divisible by* 2016*.*

Solution. Consider 1009 "holes," labeled 0, 1, ..., 1008. Assign the 1010 integers to the holes as follows. Given any integer a, there exists a unique hole label b such that either $a \equiv b \pmod{2016}$ or $a \equiv -b \pmod{2016}$; assign the integer to the corresponding hole. By pigeonhole principle, two integers out of our 1010, say a_1 and a_2, are assigned to the same hole. Then $a_1 \equiv a_2 \pmod{2016}$ or $a_1 \equiv -a_2 \pmod{2016}$. In the first case $2016 \mid (a_1 - a_2)$ and in the second case $2016 \mid (a_1 + a_2)$.