UMTYMP Advanced Topics; Fall 2016

Homework 8 (due 11/01/16)

Problem 1. *Assume that the Product Rule for formal derivatives of power series holds (we will eventually prove it in lecture). Prove that for a formal power series $F(x)$ with $F(0) \neq 0$,*

$$\left(\frac{1}{F(x)}\right)' = -\frac{F'(x)}{(F(x))^2}.$$

Problem 2. *Find an explicit formula for the exponential generating functions of the following sequences (each defined for $n \geqslant 0$):*

(1) $a_n = n$,
(2) $a_n = n^2$,
(3) $a_n = 5 \cdot n^2 + 7 \cdot n$,
(4) $a_n = 3^n$.
(5) $a_n = 5 \cdot 3^n + 7 \cdot 5^n$,

Problem 3. *In each part, a sequence $\{a_n\}_{n \geqslant 0}$ satisfies the given recurrence relation. Find the ordinary power series generating function of the sequence.*

(1) $a_{n+1} = 3a_n + 2$ $(n \geqslant 0; a_0 = 0)$
(2) $a_{n+2} = 2a_{n+1} - a_n$ $(n \geqslant 0; a_0 = 0; a_1 = 1)$

Problem 4. *A* derangement *is a permutation without fixed points (i.e. no i is in i-th place in the one-line notation). Find a formula for the number of derangements in S_n.*

Problem 5. *Let $b(n,k)$ be the number of permutations of n that have length k. Now $b(n,k)$ is defined for all $n \in \mathbb{Z}^{>0}$ and all $k \in \mathbb{Z}$, but equal to 0 if $k < 0$ or $k > \binom{n}{2}$. Find a formula for the generating function $B_n(x) = \sum_{k \in \mathbb{Z}} b(n,k)x^k$.*