Math 5705 (Enumerative Combinatorics). Fall 2017.

Homework 5 (due 10/18/17)

The next series of four problems develops the theory of even and odd permutations.

Definition. A permutation of $[n]$ is called a *transposition* if it has cycle type $(n-2, 1, 0, \ldots, 0)$, i.e. if it is a 2-cycle.

Problem 1. *Let $p = p_1 p_2 \ldots p_n$ be a permutation (in one-line notation). For $1 \leqslant i \leqslant n-1$, let s_i be the transposition $(i, i+1)$(so $s_1 = (12)$, $s_2 = (23)$, etc.)*

(1) *What is the one-line notation for ps_i?*
(2) *What is the one-line notation for $s_i p$? (this may involve cases)*

Definition. Let $p = p_1 p_2 \ldots p_n$ be a permutation. An *inversion* of p is an ordered pair (p_i, p_j) such that $i < j$ and $p_i > p_j$. For example, the permutation 314526 has inversions $(3, 1)$, $(3, 2)$, $(4, 2)$, and $(5, 2)$. The number of inversions of a permutation p is known as the *(Coxeter) length* of the permutation and is denoted $\ell(p)$.

Definition. A permutation is called *even* if it has an even number of inversions and *odd* if it has an odd number of inversions.

Problem 2. *For a positive integer n, give a combinatorial proof that there are precisely $n!/2$ even permutations of $[n]$.*

Problem 3. *Fix a positive integer n; let $S = \{(12), (23), (34), \ldots, (n-1, n)\} \subset S_n$ be the set of transpositions of adjacent elements. Prove that every $p \in S_n$ can be represented as a product of $\ell(p)$ (not necessarily distinct) elements of S.*

Hint: Try proving that we can multiply p by some $\ell(p)$ elements of S

Problem 4. *Suppose a permutation p is written as a product of k transpositions. Prove that p is even if and only if k is even.*

Now it is easy to see that a permutation is even if and only if its length is even. Also, it is easy to see that multiplying two permutations of the same parity yields an even permutation, while multiplying two permutations of different parity yields an odd permutation.

Problem 5. *Suppose G is a group. Prove that if for every $a \in G$, we have $a^2 = 1$, then G is commutative (i.e. for every $a, b \in G$ we have $a \cdot b = b \cdot a$).*

Problem 6. *Suppose $2n$ people are sitting in a circle. How many ways are there to split them into pairs if no two people sitting next to each other are allowed to form a pair?*